

A New Upper Bound for Max-2-SAT: A Graph-Theoretic Approach

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Abstract. In MAXSAT, we ask for an assignment which satisfies the maximum number of clauses for a boolean formula in CNF. We present an algorithm yielding a run time upper bound of $\mathcal{O}^*(2^{\frac{K}{6.2158}})$ for MAX-2-SAT (each clause contains at most 2 literals), where K is the number of clauses. The run time has been achieved by using heuristic priorities on the choice of the variable on which we branch. The implementation of these heuristic priorities is rather simple, though they have a significant effect on the run time. Also the analysis uses a non-standard measure.

1 Introduction

Our Problem. MAXSAT is an optimization version of the well-known decision problem SAT: given a boolean formula in CNF, we ask for an assignment which satisfies the maximum number of clauses. The applications for MAXSAT range over such fields as combinatorial optimization, artificial intelligence and database-systems as mentioned in [5]. We put our focus on MAX-2-SAT, where every formula is constrained to have at most two literals per clause, to which problems as MAXIMUM CUT and MAXIMUM INDEPENDENT SET are reducible. Therefore MAX-2-SAT is \mathcal{NP} -complete.

Results So Far. The best published upper bound of $\mathcal{O}^*(2^{\frac{K}{5.88}})$ has been achieved by Kulikov and Kutzov in [6] consuming only polynomial space. They build up their algorithm on the one of Kojevnikov and Kulikov [5] who were the first who used a non-standard measure yielding a run time of $\mathcal{O}^*(2^{\frac{K}{5.5}})$. If we measure the complexity in the number n of variables the current fastest algorithm is the one of R. Williams [10] having run time $\mathcal{O}^*(2^{\frac{\omega}{3}n})$, where $\omega < 2.376$ is the matrix-multiplication exponent. A drawback of this algorithm is its requirement of exponential space. Scott and Sorkin [9] presented a $\mathcal{O}^*(2^{1-\frac{1}{d+1}n})$ -algorithm consuming polynomial space, where d is the average degree of the variable graph. MAX-2-SAT has also been studied with respect to approximation [3,7] and parameterized algorithms [1,2].

Our Results. The major result we present is an algorithm solving MAX-2-SAT in time $\mathcal{O}^*(2^{\frac{K}{6.2158}})$. Basically it is a refinement of the algorithm in [5], which also in turn builds up on the results of [1]. The run time improvement is twofold. In [5] an upper bound of $\mathcal{O}^*(1.1225^n)$ is obtained if the variable graph is cubic.

Here n denotes the number of variables. We could improve this to $\mathcal{O}^*(1.11199^n)$ by a more accurate analysis. Secondly, in the case where the maximum degree of the variable graph is four, we choose a variable for branching according to some heuristic priorities. These two improvements already give a run time of $\mathcal{O}^*(2^{\frac{K}{6.1489}})$. Moreover we like to point out that these heuristic priorities can be implemented such that they only consume $\mathcal{O}(n)$ time. The authors of [6] improve the algorithm of [5] by having a new branching strategy when the variable graph has maximum degree five. Now combining our improvements with the ones from [6] gives the claimed run time.

Basic Definitions and Terminology. Let $V(F)$ be the set of variables of a given boolean formula F . For $v \in V(F)$ by \bar{v} we denote the negation of v . If v is set, then it will be assigned the values **true** or **false**. By the word *literal*, we refer to a variable or its negation. A *clause* is a disjunction of literals. We consider formulas in *conjunctive normal form (CNF)*, that is a conjunction of clauses. We allow only 1- and 2-clauses, i.e., clauses with at most two literals. The weight of v , written $\#_2(v)$, refers to the number of 2-clauses in which v or \bar{v} occurs. For a set $U \subseteq V(F)$ we define $\#_2(U) := \sum_{u \in U} \#_2(u)$. If v or \bar{v} occurs in some clause C we write $v \in C$. A set A of literals is called *assignment* if for every $v \in A$ it holds that $\bar{v} \notin A$. Loosely speaking if $l \in A$ for a literal l , then l receives the value **true**. We allow the formula to contain truth-clauses of the form $\{\mathcal{T}\}$ that are always satisfied. Furthermore, we consider a MAX-2-SAT instance as multiset of clauses. A $x \in V(F)$ is a *neighbor* of v , written $x \in N(v)$, if they occur in a common 2-clause. Let $N[v] := N(v) \cup \{v\}$. The *variable graph* $G_{var}(V, E)$ is defined as follows: $V = V(F)$ and $E = \{\{u, v\} \mid u, v \in V(F), u \in N(v)\}$. Observe that G_{var} is a undirected multigraph and that it neglects clauses of size one. We will not distinguish between the words “variable” and “vertex”. Every variable in a formula corresponds to a vertex in G_{var} and vice versa. By writing $F[v]$, we mean the formula which emerges from F by setting v to **true** the following way: First, substitute all clauses containing v by $\{\mathcal{T}\}$, then delete all occurrences of \bar{v} from any clause and finally delete all empty clauses from F . $F[\bar{v}]$ is defined analogously: we set x to **false**.

2 Reduction Rules & Basic Observations

We state well-known reduction rules from previous work [1,5]:

- RR-1** Replace any 2-clause C with $l, \bar{l} \in C$, for a literal l , with $\{\mathcal{T}\}$.
- RR-2** If for two clauses C, D and a literal l we have $C \setminus \{l\} = D \setminus \{\bar{l}\}$, then substitute C and D by $C \setminus \{l\}$ and $\{\mathcal{T}\}$.
- RR-3** A literal l occurring only positively (negatively, resp.) is set to **true** (**false**).
- RR-4** If \bar{l} does not occur in more 2-clauses than l in 1-clauses, such that l is a literal, then set l to **true**.
- RR-5** Let x_1 and x_2 be two variables, such that x_1 appears at most once in another clause without x_2 . In this case, we call x_2 the *companion* of x_1 . **RR-3** or **RR-4** will set x_1 in $F[x_2]$ to α and in $F[\bar{x}_2]$ to β , where $\alpha, \beta \in \{\text{true}, \text{false}\}$. Depending on α and β , the following actions will be carried out:

- If $\alpha = \text{false}$, $\beta = \text{false}$, set x_1 to **false**.
- If $\alpha = \text{true}$, $\beta = \text{true}$, set x_1 to **true**.
- If $\alpha = \text{true}$, $\beta = \text{false}$, substitute every occurrence of x_1 by x_2 .
- If $\alpha = \text{false}$, $\beta = \text{true}$, substitute every occurrence of x_1 by \bar{x}_2 .

From now on we will only consider reduced formulas F . This means that to a given formula F we apply the following procedure: **RR-i** is always applied before **RR-i+1**, each reduction rule is carried out exhaustively and after **RR-5** we start again with **RR-1** if the formula changed. A formula for which this procedure does not apply will be called *reduced*. Concerning the reduction rules we have the following lemma [5]:

- Lemma 1.** 1. If $\#_2(v) = 1$, then v will be set.
 2. For any $u \in V(F)$ in a reduced formula with $\#_2(u) = 3$ we have $|N(v)| = 3$.
 3. If the variables a and x are neighbors and $\#_2(a) = 3$, then in at least one of the formulas $F[x]$ and $F[\bar{x}]$, the reduction rules set a .

We need some auxiliary notions: A sequence of distinct vertices $a_1, v_1, \dots, v_j, a_2$ ($j \geq 0$) is called *lasso* if $\#_2(v_i) = 2$ for $1 \leq i \leq j$, $a_1 = a_2$, $\#_2(a_1) \geq 3$ and $G_{var}[a_1, v_1, \dots, v_j, a_2]$ is a cycle. A *quasi-lasso* is a lasso with the difference that $\#_2(v_j) = 3$. A lasso is called *3-lasso* (resp. *4-lasso*) if $\#_2(a_1) = 3$ ($\#_2(a_1) = 4$, resp.). *3-quasi-lasso* and *4-quasi-lasso* are defined analogously.

- Lemma 2.** 1. Let $v, u, z \in V(F)$ be pairwise distinct with $\#_2(v) = 3$ such that there are clauses C_1, C_2, C_3 with $u, v \in C_1, C_2$ and $v, z \in C_3$. Then either v is set or the two common edges of u and v will be contracted in G_{var} .
 2. The reduction rules delete the variables v_1, \dots, v_j of a lasso (quasi-lasso, resp.) and the weight of a_1 drops by at least two (one, resp.).

Proof. 1. If v is not set it will be substituted by u or \bar{u} due to **RR-5**. The emerging clauses C_1, C_2 will be reduced either by **RR-1** or become 1-clauses. Also we have an edge between u and z in G_{var} as now $u, z \in C_3$.
 2. We give the proof by induction on j . In the lasso case for $j = 0$, there must be a 2-clause $C = \{a_1, \bar{a}_1\}$, which will be deleted by **RR-1**, so that the initial step is shown. So now $j > 0$. Then on any v_i , $1 \leq i \leq j$, we can apply **RR-5** with any neighbor as companion, so, w.l.o.g., it is applied to v_1 with a_1 as companion. **RR-5** either sets v_1 , then we are done with Lemma 1.1, or v_1 will be substituted by a_1 . By applying **RR-1**, this leads to the lasso $a_1, v_2, \dots, v_j, a_2$ in G_{var} and the claim follows by induction. In the quasi-lasso case for $j = 0$, the arguments from above hold. For $j = 1$, item 1. is sufficient. For $j > 1$, the induction step from above also applies here. \square

3 The Algorithm

We set $d_i(F) := |\{x \in V(F) \mid \#_2(x) = i\}|$. To measure the run time, we choose a non standard measure approach with the measure γ defined as follows:

$$\gamma(F) = \sum_{i=3}^n \omega_i \cdot d_i(F) \text{ with } \omega_3 = 0.94165, \omega_4 = 1.80315, \omega_i = \frac{i}{2} \text{ for } i \geq 5.$$

Clearly, $\gamma(F)$ never exceeds the number of clauses K in the corresponding formula. So, by showing an upper bound of $c^{\gamma(F)}$ we can infer an upper bound c^K . We set $\Delta_3 := \omega_3$, $\Delta_i := \omega_i - \omega_{i-1}$ for $i \geq 4$. Concerning the ω_i 's we have $\Delta_i \geq \Delta_{i+1}$ for $i \geq 3$ and $\omega_4 \geq 2 \cdot \Delta_4$. The algorithm presented in this paper proceeds as follows: After applying the above-mentioned reduction rules exhaustively, it will branch on a variable v . That is, we will reduce the problem to the two formulas $F[v]$ and $F[\bar{v}]$. In each of the two branches, we must determine by how much the original formula F will be reduced in terms of $\gamma(F)$. Reduction in $\gamma(F)$ can be due to branching on a variable or to the subsequent application of reduction rules. By an (a_1, \dots, a_ℓ) -branch, we mean that in the i -th branch $\gamma(F)$ is reduced by at least a_i . The i -th component of a branch refers to the search tree evolving from the i -th branch (i.e., a_i). By writing $(\{a_1\}^{i_1}, \dots, \{a_\ell\}^{i_\ell})$ -branch we mean a $(a_1^1, \dots, a_1^{i_1}, \dots, a_\ell^1, \dots, a_\ell^{i_\ell})$ -branch where $a_j^s = a_j$ with $1 \leq s \leq i_j$. A (a_1, \dots, a_ℓ) -branch *dominates* a (b_1, \dots, b_ℓ) -branch if $a_i \geq b_i$ for $1 \leq i \leq \ell$.

Heuristic Priorities If the maximum degree of G_{var} is four, variables v with $\#_2(v) = 4$ will be called *limited* if there is another variable u appearing with v in two 2-clauses (i.e., we have two edges between v and u in G_{var}). We call such u, v a *limited pair*. Note that also u is limited and that at this point by **RR-5** no two weight 4 variables can appear in more than two clauses together. We call u_1, \dots, u_ℓ a *limited sequence* if $\ell \geq 3$ and u_i, u_{i+1} with $1 \leq i \leq \ell - 1$ are limited pairs. A *limited cycle* is a limited sequence with $u_1 = u_\ell$. To obtain an asymptotically fast algorithmic behavior we introduce heuristic priorities (**HP**), concerning the choice of the variable used for branching.

1. Choose any v with $\#_2(v) \geq 7$.
2. Choose any v with $\#_2(v) = 6$, preferably with $\#_2(N(v)) < 36$.
3. Choose any v with $\#_2(v) = 5$, preferably with $\#_2(N(v)) < 25$.
4. Choose any unlimited v with $\#_2(v) = 4$ and a limited neighbor.
5. Choose the vertex u_1 in a limited sequence or cycle.
6. Pick a limited pair u_1, u_2 . Let $c \in N(u_1) \setminus \{u_2\}$ with $s(c) := |(N(c) \cap (N(u_1)) \setminus \{c, u_1\})|$ maximal. If $s(c) > 1$, then choose the unique vertex in $N(u_1) \setminus \{u_2, c\}$, else choose u_1 .
7. From $Y := \{v \in V(F) \mid \#_2(v) = 4, \exists z \in N(v) : \#_2(z) = 3 \wedge N(z) \not\subseteq N(v)\}$ choose v , preferably such that $\#_2(N(v))$ is maximal.
8. Choose any v , with $\#_2(v) = 4$, preferably with $\#_2(N(v)) < 16$.
9. Choose any v , with $\#_2(v) = 3$, such that there is $a \in N(v)$, which forms a triangle a, b, c and $b, c \notin N[v]$ (we say v has *pending triangle* a, b, c).
10. Choose any v , such that we have a $(6\omega_3, 8\omega_3)$ - or a $(4\omega_3, 10\omega_3)$ -branch.

From now on v denotes the variable picked according to **HP**.

Key Ideas The main idea is to have some priorities on the choice of a weight 4 variable such that the branching behavior is beneficial. For example limited variables tend to be unstable in the following sense: If their weight is decreased due to branching they will be reduced due to Lemma 2.1. This means we can get an amount of ω_4 instead of Δ_4 . In a graph lacking limited vertices we want a variable v with a weight 3 neighbor u such that $N(u) \not\subseteq N(v)$. In the branch

Algorithm 1 An algorithm for solving MAX-2-SAT.

Procedure: SolMax2Sat(F)

- 1: Apply SolMax2Sat on every component of G_{var} separately.
 - 2: Apply the reduction rules exhaustively to F .
 - 3: Search exhaustively on any sub-formula being a component of at most 9 variables.
 - 4: **if** $F = \{T\} \dots \{T\}$ **then**
 - 5: **return** $|F|$
 - 6: **else**
 - 7: Choose a variable v according to **HP**.
 - 8: **return** $\max\{\text{SolMax2SAT}(F[v]), \text{SolMax2Sat}(F[\bar{v}])\}$.
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on v where u is set (Lemma 1.3) we can gain some extra reduction (at least Δ_4) from $N(u) \setminus N(v)$. If we fail to find a variable according to priorities 5-7 we show that either v as four weight 4 variables and that the graph is 4-regular, or otherwise we have two distinct situations which can be handled quite efficiently. Further, the most critical branches are when we have to choose v such that all variables in $N[v]$ have weight ω_i . Then the reduction in $\gamma(F)$ is minimal (i.e., $\omega_i + i \cdot \Delta_i$). We analyze this regular case together with its immediate preceding branch. Thereby we prove a better branching behavior compared to a separate analysis. In [9] similar ideas were used for MAX-2-CSP. We are now ready to present our algorithm, see Alg. 1. Reaching step 7 we can rely on the fact that G_{var} has at least 10 vertices. We call this the *small component property (scp)* which is crucial for some cases of the analysis.

4 The Analysis

In this section we investigate the cases when we branch on vertices picked according to items 1-10 of **HP**. For each item we will derive a branching vector which upper bounds this case in terms of K . In the rest of this section we show:

Theorem 1. *Algorithm 1 has a run time of $\mathcal{O}^*(2^{\frac{K}{6.1489}})$.*

4.1 G_{var} has Minimum Degree Four

Priority 1 If $\#_2(v) \geq 7$, we first obtain a reduction of ω_7 because v will be deleted. Secondly, we get an amount of at least $7 \cdot \Delta_7$ as the weights of v 's neighbors each drops by at least one and we have $\Delta_i \geq \Delta_{i+1}$. Thus, γ is reduced by at least 7 in either of the two branches (i.e., we have a $(\{7\}^2)$ -branch).

Regular Branches We call a branch *h-regular* if we branch on a variable v such that for all $u \in N[v]$ we have $\#_2(u) = h$. We will handle those in a separate part. During our considerations a 4-regular branch will have exactly four neighbors as otherwise this situation is handled by priority 4 of **HP**. The following subsections handle *non-regular* branches, which means that we can find a $u \in N(v)$ with $\#_2(u) < \#_2(v)$. Note that we already handled *h-regular* branches for $h \geq 7$.

Priorities 2 and 3 Choosing $v \in V(F)$ with $\#_2(v) = 6$ there is a $u \in N(v)$ with $\#_2(u) \leq 5$ due to non-regularity. Then by deletion of v , there is a reduction by ω_6 and another of at least $5\Delta_6 + \Delta_5$, resulting from the dropping weights of the neighbors. Especially, the weight of u must drop by at least Δ_5 . This leads to a $(\{6.19685\}^2)$ -branch. If $\#_2(v) = 5$, the same observations as in the last choice lead to a reduction of at least $\omega_5 + 4 \cdot \Delta_5 + \Delta_4$. Thus we have a $(\{6.1489\}^2)$ -branch.

Priority 4 Let $u_1 \in N(v)$ be the limited variable. u_1 forms a limited pair with some u_2 . After branching on v , the variable u_1 has weight at most 3. At this point, u_1 appears only with one other variable z in a 2-clause. Then, **RR-5** is applicable to u_1 with u_2 as its companion. According to Lemma 2.1, either u_1 is set or the two edges of u_1 and u_2 will be contracted. In the first case, we receive a total reduction of at least $3\omega_4 + 2\Delta_4$, in the second of at least $2\omega_4 + 4\Delta_4$. Thus, a proper estimate is a $(\{2\omega_4 + 4\Delta_4\}^2)$ -branch, i.e., a $(\{7.0523\}^2)$ -branch.

Priority 5 If u_1, \dots, u_ℓ is a limited cycle, then $\ell \geq 10$ due to *scp*. By **RR-5** this yields a $(10\omega_4, 10\omega_4)$ -branch. If u_1, \dots, u_ℓ is a limited sequence, then due to priority 4 the neighbors of u_1, u_ℓ lying outside the sequence have weight 3. By **RR-5** the branch on u_1 is a $(\{3\omega_4 + 2\omega_3\}^2)$ -branch, i.e., a $(\{7.29275\}^2)$ -branch.

Priority 6 At this point every limited variable u_1 has two neighboring variables y, z with weight 3 and a limited neighbor u_2 with the same properties (due to priorities 4 and 5). We now examine the local structures arising from this fact and by the values of $|N(y) \setminus N(u_1)|$ and $|N(z) \setminus N(u_1)|$.

1. We rule out $|N(y) \setminus N(u_1)| = |N(z) \setminus N(u_1)| = 0$ due to *scp*.
2. $|N(y) \setminus N(u_1)| = 0, |N(z) \setminus N(u_1)| = 1$: Then, $N(y) = \{u_2, z, u_1\}, N(u_2) = \{u_1, y, s_1\}$ and $N(z) = \{u_1, y, s_2\}$, see Figure 1(a). In this case we branch on z as $s(y) > 0$ and $s(y) > s(z)$. Then due to **RR-5** y and u_1 disappear; either by being set or replaced. Thereafter due to **RR-1** and Lemma 1.1 u_2 will be set. Additionally we get an amount of $\min\{2\Delta_4, \omega_4, \omega_3 + \Delta_4\}$ from s_1, s_2 . This depends on whether $s_1 \neq s_2$ or $s_1 = s_2$ and in the second case on the weight of s_1 . If $\#_2(s_1) = 3$ we get a reduction of $\omega_3 + \Delta_4$ due to setting s_1 . In total we have at least a $(\{2\omega_4 + 2\omega_3 + 2\Delta_4\}^2)$ -branch. Analogous is the case $|N(y) \setminus N(u_1)| = 1, |N(z) \setminus N(u_1)| = 0$.
3. $|N(y) \setminus N(u_1)| = 1, |N(z) \setminus N(u_1)| = 1$: Here two possibilities occur:
 - (a) $N(y) = \{u_1, u_2, s_1\}, N(z) = \{u_1, u_2, s_2\}, N(u_2) = \{u_1, y, z\}$, see Figure 1(b): Then w.l.o.g., we branch on z . Similarly to item 2. we obtain a $(\{2\omega_4 + 2\omega_3 + 2\Delta_4\}^2)$ -branch.
 - (b) $N(y) = \{u_1, z, s_1\}, N(z) = \{u_1, y, s_2\}$, see Figure 1(c): W.l.o.g., we branch on z . Basically we get a total reduction of $\omega_4 + 2\omega_3 + 2\Delta_4$. That is $2\omega_3$ from y and z , ω_4 from u_1 and $2\Delta_4$ from s_2 and u_2 . In the branch where y is set (Lemma 1.3) we additionally get Δ_4 from s_1 and ω_4 from u_2 as it will disappear (Lemma 1.2). This is a $(2\omega_4 + 2\omega_3 + 2\Delta_4, \omega_4 + 2\omega_3 + 2\Delta_4)$ -branch.
4. $|N(y) \setminus N(u_1)| = 1, |N(z) \setminus N(u_1)| = 2$, see Figure 1(d): We branch on z yielding a $(\{2\omega_4 + 2\omega_3 + 2\Delta_4\}^2)$ -branch. Analogous is the case $|N(y) \setminus N(u_1)| = 2, |N(z) \setminus N(u_1)| = 1$.

5. $|N(y) \setminus N(u_1)| = 2, |N(z) \setminus N(u_1)| = 2$: In this case we chose u_1 for branching. Essentially we get a reduction of $2\omega_4 + 2\omega_3$. In the branch setting z we receive an extra amount of $2\Delta_4$ from z 's two neighbors outside $N(u_1)$. Hence we have a $(2\omega_4 + 2\omega_3 + 2\Delta_4, 2\omega_4 + 2\omega_3)$ -branch.

We have at least a $(2\omega_4 + 2\omega_3 + 2\Delta_4, \omega_4 + 2\omega_3 + 2\Delta_4)$ -branch, i.e., a $(7.2126, 5.40945)$ -branch.

Priority 7 We need further auxiliary notions: A *3-path* (*4-path*, resp.) for an unlimited weight 4 vertex v is a sequence of vertices $u_0 u_1 \dots u_l u_{l+1}$ ($u_0 u_1 \dots u_l$, resp.) forming a path, such that $1 \leq l \leq 4$ ($2 \leq l \leq 4$, resp.), $u_i \in N(v)$ for $1 \leq i \leq l$, $\#_2(u_i) = 3$ for $1 \leq i \leq l$ ($\#_2(u_i) = 3$ for $1 \leq i \leq l-1, \#_2(u_l) = 4$, resp.) and $u_0, u_{l+1} \notin N(v)$ ($u_0 \notin N(v)$, resp.). Due to the absence of limited vertices, every vertex v , chosen due to priority 7, must have a 3- or 4-path.

3-path If $u_0 \neq u_{l+1}$ we basically get a reduction of $\omega_4 + l\omega_3 + (4-l)\Delta_4$. In the branch where u_1 is set, $u_2 \dots u_l$ will be also set due to Lemma 1.1. Therefore, we gain an extra amount of at least $2\Delta_4$ from u_0 and u_{l+1} , leading to a $(\omega_4 + l \cdot \omega_3 + (6-l)\Delta_4, \omega_4 + l \cdot \omega_3 + (4-l)\Delta_4)$ -branch.

If $u_0 = u_{l+1}$ then in $F[v]$ and in $F[\bar{v}]$, $u_0 u_1 \dots u_l u_{l+1}$ is a lasso. So by Lemma 2.2, u_1, \dots, u_l are deleted and the weight of u_0 drops by 2. If $\#_2(u_0) = 4$ this yields a reduction of $l \cdot \omega_3 + \omega_4$. If $\#_2(u_0) = 3$ the reduction is $(l+1) \cdot \omega_3$ but then u_0 is set. It is not hard to see that this yields a bonus reduction of Δ_4 (see Appendix A). Thus, we have a $(\{\omega_4 + (l+1) \cdot \omega_3 + (5-l)\Delta_4\}^2)$ -branch.

4-path We get an amount of $\omega_4 + (l-1)\omega_3 + (5-l)\Delta_4$ by deleting v . In the branch where u_1 is set we get a bonus of Δ_4 from u_0 . Further u_l will be deleted completely. Hence we have a $(2\omega_4 + (l-1)\omega_3 + (5-l)\Delta_4, \omega_4 + (l-1)\omega_3 + (5-l)\Delta_4)$ -branch.

The first branch is worst for $l = 1$, the second and third for $l = 2$ (as $l = 1$ is impossible). Thus, we have $(\{7.2126\}^2)$ -branch for the second and a $(7.0523, 5.3293)$ -branch for the first and third case which is sharp.

Priority 8 If we have chosen a variable v with $\#_2(v) = 4$ according to priority 8, such that $\#_2(N(v)) < 16$, then we have two distinct situations. By branching on v , we get at least a $(\{2\omega_4 + 2\omega_3 + 2\Delta_4\}^2)$ -branch. (See Appendix B).

The 4- 5- and 6-regular case The part of the algorithm when we branch on variables of weight $h \neq 4$ will be called *h-phase*. Branching according to priorities 4-8 is the *4-phase*, according to priorities 9 and 10 the *3-phase*. In the following we have $4 \leq h \leq 6$. Any *h-regular* branch which was preceded by a branch from the $(h+1)$ -phase can be neglected. This situation can only occur once on each path from the root to a leaf in the search tree. Hence, the run time is only affected by a constant multiple. We now classify *h-regular* branches: An *internal h-regular branch* is a *h-regular* branch such that another *h-regular* branch immediately follows in the search tree in at least one component. A *final h-regular branch* is a *h-regular* branch such that no *h-regular* branch immediately succeeds in either of the components. When we are forced to do an *h-regular* branch, then according to **HP** the whole graph must be *h-regular* at this point.

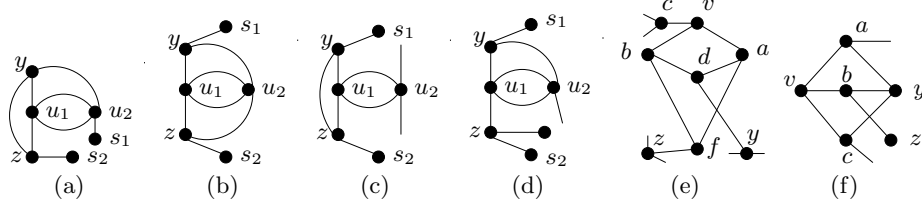


Fig. 1.

Observation 2. *If a branch is followed by a h -regular branch in one component, say in $F[v]$, then in $F[v]$ any $u \in V(F)$ with $\#_2(u) < h$ will be reduced.*

Due to Observation 2 every vertex in $N(v)$ must be completely deleted in $F[v]$.

Proposition 1. $\mathcal{O}^*(1.1088^K)$ upper bounds any internal h -regular branch.

Proof. By Observation 2 for $h = 4$ this yields at least a $(5\omega_4, \omega_4 + 4\Delta_4)$ -branch as v must have 4 different weight 4 neighbors due to **HP**. If both components are followed by an h -regular branch we get a total reduction of $5\omega_4$ in both cases. The same way we can analyze internal 5- and 6-regular branches. This yields $(3\omega_5, \omega_5 + 5\Delta_5)$ -, $(\{3\omega_5\}^2)$ -, $(3\omega_6, \omega_6 + 6\Delta_6)$ - and $(\{3\omega_6\}^2)$ -branches as for any $v \in V(F)$ we have $|N(v)| \geq 2$. \square

We now analyze a final h -regular $(\{b\}^2)$ -branch with its preceding (a_1, a_2) -branch. The final h -regular branch might follow in the first, the second or both components of the (a_1, a_2) -branch. So, the *combined analysis* would be a $(\{a_1 + b\}^2, a_2)$ -, a $(a_1, \{a_2 + b\}^2)$ - and a $(\{a_1 + b\}^2, \{a_2 + b\}^2)$ -branch.

Proposition 2. *Any final h -regular branch ($h \in \{5, 6\}$) considered together with its preceding branch can be upper bounded by $\mathcal{O}^*(1.1172^K)$.*

Proof. We will apply a combined analysis for both branches. Due to Observation 2 $N(v)$ will be deleted in the corresponding component of the preceding branch. Due to Appendices C.1 and C.2 the least amount we can get by deleting $N(v)$ is $\omega_5 + \omega_4$ in case $h = 5$ and $\omega_6 + \omega_4$ in case $h = 6$. Hence, we get four different branches: A $(\{3\omega_5 + \omega_4 + 5\Delta_5\}^2, \omega_5 + 5\Delta)$ -, a $(\{3\omega_6 + \omega_4 + 6\Delta_6\}^2, \omega_6 + 6\Delta)$ -, a $(\{3\omega_5 + \omega_4 + 5\Delta_5\}^4)$ - and a $(\{3\omega_6 + \omega_4 + 6\Delta_6\}^4)$ -branch, respectively. \square

Proposition 3. *Any final 4-regular branch considered with its preceding branch can be upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.1489}}) \approx \mathcal{O}^*(1.11933^K)$.*

Proof. We must analyze a final 4-regular branch together with any possible predecessor. These are all branches derived from priorities 4-8. See Appendix D for omitted cases.

Internal 4-regular branch The two corresponding branches are a $(\{6\omega_4 + 4\Delta_4\}^2, \omega_4 + 4\Delta_4)$ -branch and a $(\{6\omega_4 + 4\Delta_4\}^4)$ -branch.

Priorities 4, 5 and 8 are all dominated by a $(\{2\omega_4 + 4\Delta_4\}^2)$ -branch. Analyzing

these cases together with a succeeding final 4-regular branch gives a $(\{3\omega_4 + 8\Delta_4\}^2, 2\omega_4 + 4\Delta_4)$ -branch and a $(\{3\omega_4 + 8\Delta_4\}^4)$ -branch.

Priority 7 Let o be the number of weight 4 vertices from $N(v)$ and the 3- or 4-path, respectively. If in one component a final 4-regular branch follows then the worst case is when $o = 0$ as any weight 4 vertex would be deleted completely and $\omega_4 > \omega_3$. On the other hand if there is a component without an immediate 4-regular branch succeeding then the worst case appears when o is maximal as $\omega_3 \geq \Delta_4$. So in the analysis we will consider for each case the particular worst case even though both together never appear.

3-path with $u_0 \neq u_{l+1}$: First if there is a weight 4 variable in $N(u)$ we have at least the following branches: a) $(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \omega_4 + \omega_3 + 3\Delta_4)$, b) $(\omega_4 + \omega_3 + 5\Delta_4, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$ and c) $(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$. Any of those is upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.1489}})$. Now suppose for all $y \in N(v)$ we have $\#_2(y) = 3$. Table 1 captures the derived branches for certain combinations. Here we will also consider the weights of u_0 and u_l . Any entry is upper bounded

$\#_2(u_0), \#_2(u_{l+1})$	left component	right component	both components
$\#_2(u_0) = 3$ $\#_2(u_{l+1}) = 3$	$(\{2\omega_4 + 6\omega_3 + 4\Delta_4\}^2, \omega_4 + 4\omega_3)$	$(\omega_4 + 6\omega_3, \{2\omega_4 + 6\omega_3 + 4\Delta_4\}^2)$	$(\{2\omega_4 + 6\omega_3 + 4\Delta_4\}^2, \{2\omega_4 + 6\omega_3 + 4\Delta_4\}^2)$
$\#_2(u_0) = 3$ $\#_2(u_{l+1}) = 4$	$(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \omega_4 + 4\omega_3)$	$(\omega_4 + 5\omega_3 + \Delta_4, \{2\omega_4 + 5\omega_3 + 4\Delta_4\}^2)$	$(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \{2\omega_4 + 5\omega_3 + 4\Delta_4\}^2)$
$\#_2(u_0) = 4$ $\#_2(u_{l+1}) = 4$	$(\{4\omega_4 + 4\omega_3 + 4\Delta_4\}^2, \omega_4 + 4\omega_3)$	$(\omega_4 + 4\omega_3 + 2\Delta_4, \{2\omega_4 + 4\omega_3 + 4\Delta_4\}^2)$	$(\{4\omega_4 + 4\omega_3 + 4\Delta_4\}^2, \{2\omega_4 + 4\omega_3 + 4\Delta_4\}^2)$

Table 1.

by $\mathcal{O}^*(2^{\frac{K}{6.1489}})$ except $\alpha)$ $(\{2\omega_4 + 6\omega_3 + 4\Delta_4\}^2, \omega_4 + 4\omega_3)$ the left upper entry and $\beta)$ $(\omega_4 + 4\omega_3 + 2\Delta_4, \{2\omega_4 + 4\omega_3 + 4\Delta_4\}^2)$ the middle entry of the last row. For $U \subseteq V(F)$ we define $E_3(U) := \{\{u, v\} \mid u \in U, \#_2(u) = 3, v \notin U\}$.

- Claim.* 1. Suppose for all $y \in Q := N(v) \cup \{u_0, u_{l+1}\}$ we have $\#_2(y) = 3$. Then there must be some $y' \in V \setminus (N(v) \cup \{u_0, u_{l+1}\})$ with $\#_2(y') = 3$.
2. Suppose for all $y \in N(v)$ we have $\#_2(y) = 3$ and $\#_2(u_0) = \#_2(u_{l+1}) = 4$. Then there must be some $y' \in V \setminus (N(v) \cup \{u_0, u_{l+1}\})$ with $\#_2(y') = 3$.

- Proof.* 1. Assume the contrary. For any $1 \leq l \leq 4$ we have $|E_3(Q \cup \{v\})| \leq 10$. Due to *scp* there is a weight 4 vertex r adjacent to some vertex in Q . Observe that we must have $r \in Y$ as either there is $u \in N(v)$ with $u \in N(r)$ and $v \notin N(r)$ or w.l.o.g. $u_0 \in N(r)$ but $u_1 \notin N(r)$. Hence, r has 4 weight 3 neighbors from Q due to the choice of v . Hence we must have $|E_3(Q \cup \{v, r\})| \leq 6$. Using the same arguments again we find some $r' \in Y$ with $|E_3(Q \cup \{v, r, r'\})| \leq 2$. Again, due to *scp* we find a $r'' \in Y$ with 4 weight 3 neighbors where at most two are from Q , a contradiction.
2. Assume the contrary. Observe that $u_0, u_{l+1} \in Y$ and due to the choice of v both have 4 weight 3 neighbors which must be from $N(v)$. From $|E_3(N[v])| \leq 8$ follows that $|E_3(N[v] \cup \{u_0, u_{l+1}\})| = 0$ which contradicts *scp*. \square

Due to the last claim and Observation 2 we have a $(\{2\omega_4 + 7\omega_3 + 4\Delta_4\}^2, \omega_4 + 4\omega_3)$ -branch for case α) and a $(\omega_4 + 4\omega_3 + 2\Delta_4, \{2\omega_4 + 5\omega_3 + 4\Delta_4\}^2)$ -branch for case β). Both are upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.1489}})$. \square

4.2 The Cubic Case

Priority 9 Observe that when we have arrived at this point, the graph G_{var} must be 3-regular and each variable has three different neighbors, due to G_{var} being reduced and due to Lemma 1.2. Also, any 3-regular graph has an even number of vertices, because we have $3n = 2m$. Thus any branching must be of the form $(2i \cdot \omega_3, 2j \cdot \omega_3)$ for some $1 \leq i, j$. Also, branching on any variable will at least result in a $(4\omega_3, 4\omega_3)$ -branch (see Lemma 1.2). Note that any $u \in N(v)$ will be either set in $F[v]$ or in $F[\bar{v}]$, due to Lemma 1.3.

Lemma 3. *Let v have a pending triangle a, b, c and $N(v) = \{a, p, q\}$. Then by branching on v , we have an $(8\omega_3, 6\omega_3)$ -branch.*

Proof. In $F[v]$ and $F[\bar{v}]$, the variables a, b, c form a 3-quasi-lasso. Hence, due to Lemma 2.2 w.l.o.g., only b remains in the reduced formula with $\#_2(b) = 2$ (Lemma 2.2). Also, in both branches, q and p are of weight two and therefore deleted. Note that $N(\{q, p\}) \cup \{q, p\} \subseteq \{v, a, b, c, q, p\}$, contradicts *scp*. Therefore, w.l.o.g., there is a variable $z \in N(q)$ such that $z \notin \{v, a, b, c, q, p\}$. So, in the branch where q is set, also z will be deleted. Thus, seven variables will be deleted. \square

Priority 10 From now on, due to **HP**, G_{var} is triangle-free and cubic. We show that if we are forced to choose a vertex v to which none of the priorities 1-9 fits, we can choose v such that we obtain either a $(6\omega_3, 8\omega_3)$ - or a $(4\omega_3, 10\omega_3)$ -branch.

Lemma 4. *Let v be a vertex in G_{var} and $N(v) = \{a, b, c\}$. Suppose that, w.l.o.g., in $F[v]$ a, b and in $F[\bar{v}]$ c will be set. Then we have a $(6\omega_3, 8\omega_3)$ -branch.*

Proof. If $|(N(a) \cap N(b)) \setminus \{v\}| \leq 1$, then by setting a and b in $F[v]$, five variables will be reduced. Together with v and c , this is a total of seven. If $|(N(a) \cap N(b)) \setminus \{v\}| = 2$, then situation 1(e) must occur (note the absence of triangles). If $z = y$ then also $z \neq c$ due to *scp*. Then in $F[v]$ due to Lemma 1.1 v, a, b, c, d, f, z will be deleted. If $z \neq y$ then v, a, b, d, f, z, y will be deleted. Together with $F[\bar{v}]$ where c is set, we have a $(6\omega_3, 8\omega_3)$ -branch. \square

Lemma 5. *If for any $v \in V(F)$ all its neighbors are set in one branch (say, in $F[v]$), we can perform a $(6\omega_3, 8\omega_3)$ - or a $(4\omega_3, 10\omega_3)$ -branch due to cubicity.*

Proof. If $|N(a, b, c) \setminus \{v\}| \geq 5$, then in $F[v]$, 9 variables are deleted, so that we have a $(4\omega_3, 10\omega_3)$ -branch. Otherwise, either one of the two following situations must occur: a) There is a variable $y \neq v$, such that $N(y) = \{a, b, c\}$, see Figure 1(f). Then branch on b . In $F[\bar{b}]$, v, y, a, c, z will disappear (due to **RR-5** and Lemma 2.1). In $F[b]$, due to setting z , additionally a neighbor $f \notin \{a, b, c, v, y\}$

of z will be deleted due to *scp*. This is a total of seven variables.

b) There are variables p, q , such that $|N(p) \cap \{a, b, c\}| = |N(q) \cap \{a, b, c\}| = 2$. The last part of Theorem 4.2 of [5] handles *b*). See also Appendix E. \square

Due to the last three lemmas, branchings according to priorities 9 and 10 are upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.1489}})$. Especially, the $(4\omega_3, 10\omega_3)$ -branch is sharp.

5 Combining Two Approaches

Kulikov and Kutzov [6] achieved a run time of $\mathcal{O}^*(2^{\frac{K}{5.88}})$. This was obtained by speeding up the 5-phase by a concept called 'clause learning'. As in our approach the 3- and 4-phase was improved we will show that if we use both strategies we can even beat our previous time bound. This means that in **HP** we substitute priority 3 by their strategy with one exception: we prefer variables v with a non weight 5 neighbor. Forced to violate this preference we do a simple branching of the form $F[v]$ and $F[\bar{v}]$. For the analysis we redefine the measure $\gamma(F)$: we set $\omega_3 = 0.9521$, $\omega_4 = 1.8320$, $\omega_5 = 2.488$ and keep the other weights. We call this measure $\tilde{\gamma}(F)$. We will reproduce the analysis of [6] briefly with respect to $\tilde{\gamma}(F)$ to show that their derived branches for the 5-phase are upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.2158}})$. It also can be checked that this is also true for the branches derived for the other phases by measuring them in terms of $\tilde{\gamma}(F)$, see Appendix F.2. Let k_{ij} denote the number of weight j variables occurring i times in a 2-clause with some $v \in V(F)$ chosen for branching. Then we must have: $k_{13} + k_{14} + k_{15} + 2k_{24} + 2k_{25} + 3k_{35} = 5$. If F' is the the formula obtained by assigning a value to v and by applying the reduction rules afterwards we have:

$$\tilde{\gamma}(F) - \tilde{\gamma}(F') \geq 5\Delta_5 + \omega_5 + (\omega_3 - \Delta_5)k_{13} + (\Delta_4 - \Delta_5)k_{14} + (\frac{\omega_4}{2} - \Delta_5)2k_{24} + (\Delta_4 - \Delta_5)k_{25} + (\frac{\omega_5}{2} - \frac{3}{2}\Delta_5)2k_{35} = 5.768 + 0.2961k_{13} + 0.2239(k_{14} + k_{25}) + 0.26 \cdot 2(k_{24} + k_{35})$$

Basically we reduce $\tilde{\gamma}(F)$ by at least $\omega_5 + 5\Delta_5$. Now the coefficients of the k_{ij} in the above equation express how the reduction grows if $k_{ij} > 0$. If $k_{13} + k_{14} + 2k_{24} + k_{25} + 2k_{35} \geq 2$ we are done as $\tilde{\gamma}(F) - \tilde{\gamma}(F') \geq 6.2158$.

If $k_{13} = 1$ and $k_{15} = 4$ then [6] stated a $(5\Delta_5 + \omega_5 + (\omega_3 - \Delta_5), 5\Delta_5 + \omega_5 + (\omega_3 - \Delta_5) + 2\Delta_5)$ -branch and for $k_{25} = 1$ and $k_{15} = 3$ a $(5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5), 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + \omega_3)$ -branch. If $k_{14} = 1$ and $k_{15} = 4$ a branching of the kind $F[v], F[\bar{v}, v_1], F[\bar{v}, \bar{v}_1, v_2, v_3, v_4, v_5]$ is applied, where $\{v_1, \dots, v_5\} = N(v)$. From this follow a $(5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5), 4\Delta_5 + \omega_5 + \Delta_4 + \omega_4 + 3\Delta_4 + \Delta_5, 5\omega_5 + \omega_4)$ - and a $(\omega_5 + 4\Delta_5 + \Delta_4, \omega_5 + 4\Delta_5 + \Delta_4 + \omega_4 + 4\Delta_5, 5\omega_5 + \omega_4 + 3\omega_3)$ -branch. This depends on whether v_1 has at least three neighbors of weight less than 5 in $F[\bar{v}]$ or not. We observed that we can get a additional reduction of Δ_5 in the third component of the first branch as $N[v]$ cannot be a component in $V(F)$ after step 3 of Alg. 1 yielding a $(4\Delta_5 + \omega_5 + \Delta_4, 5\Delta_5 + \omega_5 + 4\Delta_4 + \omega_4, 5\omega_5 + \omega_4 + \Delta_5)$ -branch. For the analysis of the 5-regular branch (i.e. $k_{15} = 5$) we refer to Appendix F.1. It proceeds the same way as in the simple version of the algorithm except that we have to take into account the newly introduced branches.

Theorem 3. MAX-2-SAT can be solved in time $\mathcal{O}^*(2^{\frac{K}{6.2158}}) \approx \mathcal{O}^*(1.118^K)$.

6 Conclusion

We presented an algorithm solving MAX-2-SAT in $\mathcal{O}^*(2^{\frac{K}{5.2158}})$, with K the number of clauses of the input formula. This is currently the end of a sequence of polynomial-space algorithms each improving on the run time: beginning with $\mathcal{O}^*(2^{\frac{K}{2.88}})$ which was achieved by [8], it was subsequently improved to $\mathcal{O}^*(2^{\frac{K}{3.742}})$ by [2], to $\mathcal{O}^*(2^{\frac{K}{5}})$ by [1], to $\mathcal{O}^*(2^{\frac{K}{5.217}})$ by [4], to $\mathcal{O}^*(2^{\frac{K}{5.5}})$ by [5] and finally to the hitherto fastest upper bound of $\mathcal{O}^*(2^{\frac{K}{5.88}})$ by [6]. Our improvement has been achieved due to heuristic priorities concerning the choice of variable for branching in case of a maximum degree four variable graph. As [6] improved the case where the variable graph has maximum degree five, it seems that the only way to speed up the generic branching algorithm is to improve the maximum degree six case. Our analysis also implies that the situation when the variable graph is regular is not that harmful. The reason for this that the preceding branch must have reduced the problem size more than expected. Thus considered together these two branches balance each other. Though the analysis is to some extent sophisticated and quite detailed the algorithm has a clear structure. The implementation of the heuristic priorities for the weight 4 variables should be a straightforward task. Actually, we have already an implementation of Alg. 1. It is still in an early phase but nevertheless the performance is promising. We are looking forward to report on these results on another occasion.

References

1. J. Gramm, E. A. Hirsch, R. Niedermeier, and P. Rossmanith. Worst-case upper bounds for MAX-2-SAT with an application to MAX-CUT. *Discrete Applied Mathematics*, 130:139–155, 2003.
2. J. Gramm and R. Niedermeier. Faster exact solutions for MAX2SAT. In *CIAC*, volume 1767 of *LNCS*, pages 174–186. Springer, 2000.
3. T. Hofmeister. An approximation algorithm for MAX-2-SAT with cardinality constraint. In *ESA*, volume 2832 of *LNCS*, pages 301–312. Springer, 2003.
4. J. Kneis, D. Mölle, S. Richter, and P. Rossmanith. Algorithms based on the treewidth of sparse graphs. In *WG*, volume 3787 of *LNCS*, pages 385–396. Springer, 2005.
5. A. Kojevnikov and A. S. Kulikov. A new approach to proving upper bounds for MAX-2-SAT. In *SODA*, pages 11–17. ACM Press, 2006.
6. A. S. Kulikov and K. Kutukov. New bounds for max-sat by clause learning. In *CSR*, volume 4649 of *LNCS*, pages 194–204. Springer, 2007.
7. M. Lewin, D. Livnat, and U. Zwick. Improved rounding techniques for the MAX 2-SAT and MAX DI-CUT problems. In *IPCO*, volume 2337 of *LNCS*, pages 67–82. Springer, 2002.
8. R. Niedermeier and P. Rossmanith. New upper bounds for maximum satisfiability. *Journal of Algorithms*, 36:63–88, 2000.
9. A. Scott and G. Sorkin. Linear-programming design and analysis of fast algorithms for Max 2-CSP. *Discrete Optimization*, 4(3-4): 260–287, 2007.
10. R. Williams. A new algorithm for optimal 2-constraint satisfaction and its implications. *Theoretical Computer Science*, 348(2-3):357–365, 2005.

A Additional Arguments Concerning 3-paths in the Non-regular Case

Proposition 4. *Let $v \in V(F)$ be chosen due to **HP** such that $\#_2(v) = 4$ and v has a 3-path of length l such that $u_0 = u_{l+1}$. Then we have at least a $(\{\omega_4 + 3\omega_3 + 3\Delta_4\}^2)$ -branch.*

Proof. In $F[v]$ and in $F[\bar{v}]$, $u_0 u_1 \dots u_l u_{l+1}$ is a lasso. So by Lemma 2.2, u_1, \dots, u_l are deleted and the weight of u_0 drops by 2. If $\#_2(u_0) = 4$ this yields a reduction of $l \cdot \omega_3 + \omega_4$. If $\#_2(u_0) = 3$ the reduction is $(l+1) \cdot \omega_3$ but then u_0 is set. If $N(u_0) \setminus N(v)$ is non-empty then we obtain a reduction of Δ_4 in addition due to setting u_0 . Otherwise there is a unique $r \in N(u_0) \setminus \{u_1, \dots, u_l\}$ with $r \in N(v) \setminus \{u_1, \dots, u_l\}$. If $\#_2(r) = 4$ we get a $(\{2\omega_4 + (l+1)\omega_3 + (3-l)\Delta_4\}^2)$ -branch. If $\#_2(r) = 3$ then r is set. As $(4-l) \leq 2$ and by applying the same arguments to r which previously were applied to u_0 we get at least a $(\{\omega_4 + (l+1) \cdot \omega_3 + (5-l)\Delta_4\}^2)$ -branch. Observe that we used the fact that $\omega_4 \geq 2\Delta_4$. \square

B Proof of the Statement in Priority 8 (Non-regular Case)

Proof. Note that when we are forced to pick a variable v according to priority 8, then either v has four neighbors of weight 4 or for every weight 3 neighbor z we have $N(z) \subseteq N(v)$. From $\#_2(N(v)) < 16$ follows that, for every weight 3 neighbor z , we have $N(z) \subseteq N(v)$ due to the choice of v according to **HP**. Let N^4 (resp. N^3) be the set of weight four (three) neighbors of v . We analyze different cases induced by $k := |N_3|$. Let $k = 1$. If $N^3 = \{b\}$, then there are vertices $a, c \in N^4$, such that $b \in N(a)$ and $b \in N(c)$. We must have $a \in N(c)$, or else a would violate our assumption. Thus, we get the situation of Figure 2(a).

Let $k = 2$. Then $N^3 = \{b, c\}$ and assume that b and c are neighbors. If $b, c \in N(a)$ for $a \in N^4$, we have situation depicted in Figure 2(b). Otherwise $b \in N(a)$ and $c \in N(d)$ for $a, d \in N^4$. But then, priority 7 applies to both a and d , which is a contradiction. In the case where b and c are not neighbors, it can be easily observed that we must have the situation in Figure 2(c), where priority 7 applies to a and d .

If $k = 3$ it is easy to verify that we must have situation 2(d) in Figure 1. But then priority 7 applies to a . If $k = 4$ then clearly $N[v]$ forms a component of five vertices which cannot appear after step 3 of Alg. 1.

In Figure 2(a) in either branch $F[v]$ or $F[\bar{v}]$, the variables a, b, c form a 3-quasi-lasso, so by Lemma 2.2 we get a reduction of $\omega_3 + 3\omega_4 + \Delta_4 = 4\omega_4$.

In Figure 2(b) in both branches the variables a, b, c form a 3-lasso, so by Lemma 2.2 b, c are deleted and a is set due to Lemma 1.1. We get a reduction of $\omega_4 + 2\omega_3$ from this. If $d \notin N(a)$ we additionally get $2\Delta_4$, otherwise ω_4 . Altogether, we reduce $\gamma(F)$ by at least $2\omega_4 + 2\omega_3 + 2\Delta_4$. \square

C Analysis of Internal 5- and 6-regular Branches

We will consider branches which are immediately followed by a h -regular branch in at least one component. In this component of the branch we can delete any variable

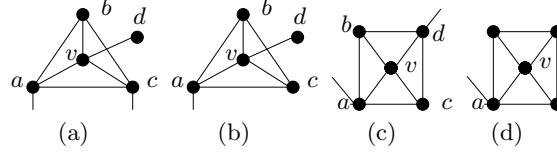


Fig. 2.

in $N(v)$ additionally due to Observation 2. The Subsections C.1 and C.2 will explore by how much we additionally can decrement $\gamma(F)$ in the corresponding component in case $h \in \{5, 6\}$. Let k_{ij} denote the number of weight j variables occurring i times in a 2-clause with some $v \in V(F)$ chosen for branching.

C.1 Internal 5-regular Branches

Proposition 5. *Let $v \in V(F)$ be the variable chosen due to **HP** such that $\#_2(v) = 5$. If this branch is followed by a 5-regular branch in one component, then we can decrement $\gamma(F)$ by at least $\omega_5 + \omega_4$ in addition to the weight of v in that component.*

Proof. According to [6] we must have the following relation:

$$k_{13} + k_{14} + k_{15} + 2k_{24} + 2k_{25} + 3k_{35} = 5 \quad (1)$$

We now have to determine an integer solution to (1) such that $\omega_3 k_{13} + \omega_4 k_{14} + \omega_5 k_{15} + \omega_4 k_{24} + \omega_5 k_{25} + \omega_5 k_{35}$ is minimal. We can assume $k_{14} = k_{15} = 0$ as we have $\omega_3 < \omega_4 < \omega_5$. For any solution violating this property we can find a smaller solution by setting $k'_{13} = k_{13} + k_{14} + k_{15}$, $k'_{14} = 0$ and $k'_{15} = 0$ and keeping the other coefficients. The same way we find that $k_{25} = 0$ must be the case as $\omega_4 < \omega_5$.

If $k_{13} \geq 2$ we set $k'_{13} = k_{13} - 2 \lfloor \frac{k_{13}}{2} \rfloor$, $k'_{24} = k_{24} + \lfloor \frac{k_{13}}{2} \rfloor$ and keep the other coefficients. By $2\omega_3 > \omega_4$ this is a smaller solution. Now suppose $k_{13} = 1$, then we have $k_{24} = 0$ in a minimal solution as $\omega_3 + \omega_4 > \omega_5$ (i.e., if $k_{24} \geq 1$ we set $k'_{13} = 0$, $k'_{24} = k_{24} - 1$ and $k'_{35} = k_{35} + 1$). But then no k_{35} could satisfy (1). Thus, we have $k_{13} = 0$. Then the only solution is $k_{24} = 1$ and $k_{35} = 1$. Hence, the minimal reduction we get from $N(v)$ is $\omega_5 + \omega_4$. \square

C.2 Internal 6-regular Branches

Proposition 6. *Let $v \in V(F)$ be the variable chosen due to **HP** such that $\#_2(v) = 6$. If this branch is followed by a 6-regular branch in one component, then we can decrement $\gamma(F)$ by at least $\omega_6 + \omega_4$ in addition to the weight of v in that component.*

Proof. In this case the following relation holds:

$$k_{13} + k_{14} + k_{15} + k_{16} + 2k_{24} + 2k_{25} + 2k_{26} + 3k_{35} + 3k_{36} + 4k_{46} = 6 \quad (2)$$

We now have to determine an integer solution to (2) such that $\omega_3 k_{13} + \omega_4 k_{14} + \omega_5 k_{15} + \omega_6 k_{16} + \omega_4 k_{24} + \omega_5 k_{25} + \omega_6 k_{26} + \omega_5 k_{35} + \omega_6 k_{36} + \omega_6 k_{46}$ is minimal. As $\omega_3 < \omega_4 < \omega_5 < \omega_6$ we conclude that $k_{1\ell} = 0$ for $4 \leq \ell \leq 6$, $k_{2\ell'} = 0$ for $5 \leq \ell' \leq 6$ and $k_{36} = 0$. We also must have $k_{13} \leq 1$ as in the section above. By $2\omega_4 > \omega_6$ we must have $k_{24} \leq 1$. By (2)

we also have $k_{35} \leq 2$ and $k_{46} \leq 1$.

If $k_{13} = 0$ the only solutions under the given restrictions are $k_{35} = 2$ and $k_{24} = 1, k_{46} = 1$. If $k_{13} = 1$ the only solution is $k_{35} = 1, k_{24} = 1$. Thus, the minimal amount we get by reduction from $N(v)$ is $\omega_6 + \omega_4$. \square

D Omitted Cases of the Analysis of the Final 4-regular Case

Proposition 7. *Any final 4-regular branch considered with its preceding branch can be upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.1489}})$.*

Proof. Here we must analyze a final 4-regular branch together with any possible predecessor. These are all branches derived from priorities 4-8: Let o be the number of weight 4 vertices from $N(v)$ or the 3- or 4-path, respectively. If in one component a final 4-regular branch follows then the worst case is when $o = 0$ as any such vertex would be deleted completely and $\omega_4 > \omega_3$. On the other hand if there is a component without an immediate 4-regular branch succeeding then the worst case appears when o is maximal as $\omega_3 \geq \Delta_4$. So in the analysis we will consider for each case the particular worst case even though both together never appear.

Priority 6 Subcases 2, 3(a) and 4 of our non-regular priority-6 analysis can be analyzed similar to priorities 4, 5 and 8. We now analyze the remaining subcases.

Subcase 1 Here we deal with small components which are directly solved without any branching. Therefore we get a $(\{3\omega_4 + 2\omega_3 + 4\Delta_4\}^2)$ -branch in the combined analysis.

Consider now cases 5 and 3(b). Let u_1, u_2 be the picked limited pair. Due to **HP** the variable u_2 has two weight 3 neighbors. Thus, if a final 4-regular branch is following in these cases we get a reduction of $2\omega_3$ in addition (with respect to the component of the branch). For both cases we derived a non symmetric branch, e.g., an (a, b) -branch with $a \neq b$. Depending whether the final 4-regular branch follows in the first, the second or both components we derive three combined branches: *a*) $(\{3\omega_4 + 4\omega_3 + 4\Delta_4\}^2, 2\omega_3 + \omega_4 + 2\Delta_4)$, *b*) $(2\omega_3 + 2\omega_4 + 2\Delta_4, \{3\omega_4 + 4\omega_3 + 4\Delta_4\}^2)$ - and *c*) $(\{3\omega_4 + 4\omega_3 + 4\Delta_4\}^2, \{3\omega_4 + 4\omega_3 + 4\Delta_4\}^2)$. As $\mathcal{O}^*(2^{\frac{K}{6.1489}})$ not proper upperbounds *a*) we need a further discussion for the two subcases. Remember that in the first component of *a*) some weight 3 neighbor t of v is set.

Subcase 3b First suppose that $N(z) \setminus (N(u_1) \cup N(u_2)) = \emptyset$ and $N(y) \setminus (N(u_1) \cup N(u_2)) = \emptyset$, see Figure 3(a). Then by either branching on y or z we get a $(\{2\omega_4 + 4\omega_3\}^2)$ -branch. In this case the combined analysis is similar to priorities 4, 5 and 8. Secondly, w.l.o.g. we have $N(z) \setminus (N(u_1) \cup N(u_2)) \neq \emptyset$, see Figure 3(b) and 3(c). In Figure 3(b) we might have picked $y = v$ or $z = v$. But observe that in both cases in the branch where the particular weight 3 neighbor t is set ($t = s$ if $v = z$ and $t = z$ if $v = y$) such that in this component a 4-regular branch follows we have a $(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \omega_4 + 2\omega_3 + 2\Delta_4)$ -branch in the combined analysis instead of *a*). If the case in Figure 3(c) matches then we have $t = s$. Then in the branch where s is set y and u_1 will be reduced due to **RR-5** and $N[u_2] \setminus \{u_1\}$ due to the fact that a 4-regular branch follows. Thus, the derived branch is the same as for the case of Figure 3(b).

Subcase 5 As the vertices in $N(u_1) \cup N(u_2)$ can not form a component w.l.o.g. we have that $N(z) \setminus (N(u_1) \cup N(u_2)) \neq \emptyset$. In this case we branch on u_1 . Now in the

branch where we set z (i.e., $z = t$) such that a 4-regular branch follows in that component we have a $(\{3\omega_4 + 5\omega_3 + 4\Delta_5\}^2, 2\omega_4 + 2\omega_3)$ -branch in the combined analysis insted of a)

Both branches replacing a) have an upper bound of $\mathcal{O}^*(2^{\frac{K}{6.1489}})$.

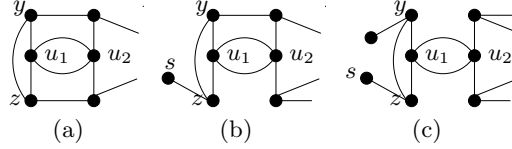


Fig. 3.

Priority 7 4-path In this case we have the following branches: a) $(\{3\omega_4 + 4\omega_3 + 4\Delta_4\}^2, \omega_4 + \omega_3 + 3\Delta_4)$, b) $(2\omega_4 + \omega_3 + 3\Delta_4, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$, c) $(\{3\omega_4 + 4\omega_3 + 4\Delta_4\}^2, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$. The cases a) and c) are not upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.1489}})$ and hence need further discussion.

Suppose there is a vertex $y \in D := N(v) \cup \{u_0, \dots, u_{l-1}\}$ with weight 4. Then by Observation 2 we have branches $a')$ $(\{4\omega_4 + 3\omega_3 + 4\Delta_4\}^2, \omega_4 + \omega_3 + 3\Delta_4)$ and $c')$ $(\{4\omega_4 + 3\omega_3 + 4\Delta_4\}^2, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$ which are both upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.1489}})$. For the remaining case we need the next proposition. For $U \subseteq V(F)$ we define $E_3(U) = \{\{u, v\} \mid u \in U, \#_2(u) = 3, v \notin U\}$.

Claim. Suppose for all $y \in D$ we have $\#_2(y) = 3$. Then there must be some $y' \in V \setminus (D \cup \{v, u_l\})$ with $\#_2(y') = 3$.

Proof. Assume the contrary. Observe that if $l \geq 3$ then $u_l \in Y$ due to $u_{l-2} \notin N(u_l)$. If $l = 2$ and $u_0 \notin N(u_2)$ then also $u_l \in Y$ holds. Let us assume this case as the other one will be treated separately.

Now due to the choice of v we have that u_l must be adjacent to v, u_{l-1} and to two further weight 3 vertices in D . Therefore and as $D \cup \{v, u_l\}$ can not be a component we have $l < 4$. Also for any $2 \leq l \leq 3$ we always have $|E_3(D \cup \{v, u_l\})| \leq 8 - 2l$. There must some weight 4 vertex $r \notin D \cup \{v, u_l\}$ adjacent to some weight 3 vertex in D as we have no small components. Note that $r \in Y$, as $v \notin N(r)$ or w.l.o.g. $u_0 \in N(r)$ but $u_1 \notin N(r)$. Due to the choice of v , r must have at least three weight 3 neighbors. Hence $l = 2$. If r has 4 weight 3 neighbors then $(D \cup \{v, u_l, r\})$ forms a component which is a contradiction. Hence, we have $|E_3(D \cup \{v, u_l, r\})| = 1$ and therefore we find again some $r' \in Y \setminus (D \cup \{v, u_l, r\})$ which is adjacent to at least 3 weight 3 vertices where at most one is from D . Thus there must be some weight 3 vertex in $V \setminus (D \cup \{v, u_l\})$, a contradiction.

Now suppose $l = 2$ and $u_0 \in N(u_2)$. Let $N(u_0) = \{z, u_1, u_2\}$. If $z \notin N(u_2)$ then $u_2 \in Y$ and the first part of the proof applies. Now suppose $z \in N(u_2)$ and $\#_2(z) = 3$. Then it follows that $z \in N(v)$ and $|E_3(\{D \cup \{v, u_2\}\})| \leq 2$. Now due to *scp* we can find an $r \in Y \setminus (D \cup \{v, u_l\})$ which is adjacent to at least three weight 3 vertices where only two can be from $D \cup \{v, u_l\}$, a contradiction.

Suppose $z \in N(u_2)$ and $\#_2(z) = 4$. As $u_1 \notin N(z)$ we have $z \in Y$. Thus, z is adjacent to the two weight 3 vertices in $N(v) \setminus \{u_2, u_1\}$. As $D \cup \{v, u_2, z\}$ is not a

component we have $|E_3(\{D \cup \{v, u_2, z\}\})| = 2$. Similarly, a contradiction follows. \square

If for all $y \in D$ we have $\#_2(y) = 3$ from the last claim and Lemma 2.2 we can derive two branches a'') ($\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \omega_4 + \omega_3 + 3\Delta_4$) and c'') ($\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2$) which are upper bounded by $\mathcal{O}^*(2^{\frac{K}{6.1489}})$.

3-path In the case of 3-path such that $u_0 = u_{l+1}$ the branch with $l = 2$ is dominated by all other choices. Since this is a $(\{7.21\}^2)$ -branch we refer to priorities 4, 5 and 8 from above. \square

E Full Proof of Lemma 5

Proof. If $|N(a, b, c) \setminus \{v\}| \geq 5$, then in $F[v]$, 9 variables are deleted, so that we have a $(4\omega_3, 10\omega_3)$ -branch. Otherwise, either one of the two following situations must occur:
a) There is a variable $y \neq v$, such that $N(y) = \{a, b, c\}$, see Figure 1(f). Then branch on b . In $F[\bar{b}]$, v, y, a, c, z will disappear (due to **RR-5** and Lemma 2.1). In $F[b]$, due to setting z , additionally a neighbor $f \notin \{a, b, c, v, y\}$ of z will be deleted as the vertices a, b, c, v, y, z do not form a component. This is a total of seven variables.

b) There are variables p, q , such that $|N(p) \cap \{a, b, c\}| = |N(q) \cap \{a, b, c\}| = 2$, see Figure 4(a) and 4(b). In $F[v]$, the variables a, b, c, p, q will be set. Then, at least 3 additional variables will be deleted (even if there are $1 \leq i < j \leq 4$ with $h_i = h_j$). Theorem 4.2 of [5] contains also an alternative proof of b). \square

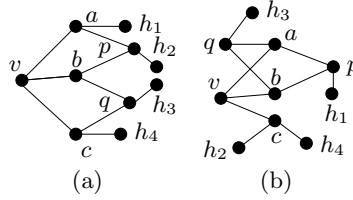


Fig. 4.

F Additional Analysis of the Combined Approach

F.1 5-regular Branches in the Combined Approach

Internal 5-regular branches yield the same recurrences as in the simple approach. Final 5-regular branches must be analyzed together with their immediate preceding branch. Thus they have to be analyzed together with the introduced branches of [6]. Table 2 captures some cases ($k_{15} = 5$; $k_{13} = 1, k_{14} = 4$; $k_{25} = 1, k_{15} = 3$). For the case $k_{14} = 1$ and $k_{15} = 4$ there are two recurrences for the branching $F[v], F[\bar{v}, v_1], F[\bar{v}, \bar{v}_1, v_2, v_3, v_4, v_5]$.

case	one component	both components	upper bound
$k_{15} = 5$	$(\{7\omega_5 + 5\Delta_5\}^2, \omega_5 + 5\Delta_5)$	$(\{7\omega_5 + 5\Delta_5\}^4)$	$\mathcal{O}^*(1.0846^K)$
$k_{13} = 1, k_{15} = 4$	$(\{6\omega_5 + \omega_3 + 5\Delta_5\}^2, \omega_5 + 4\Delta_5 + \omega_3)$	$\{6\omega_5 + \omega_3 + 5\Delta_5\}^4$	$\mathcal{O}^*(1.0878^K)$
$k_{25} = 1, k_{15} = 3$	$(\{6\omega_5 + 5\Delta_5\}^2, \omega_5 + 3\Delta_5 + (\omega_5 - \omega_3))$	$(\{6\omega_5 + 5\Delta_5\}^4)$	$\mathcal{O}^*(1.0914^K)$

Table 2.

The first recurrence assumes that v_1 has at least three neighbor of weight less than five in $F[\bar{v}]$: **(A)** $(5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5), 4\Delta_5 + \omega_5 + \Delta_4 + \omega_4 + 3\Delta_4 + \Delta_5, 5\omega_5 + \omega_4 + \Delta_5)$. The other **(B)** $(\omega_5 + 4\Delta_5 + \Delta_4, \omega_5 + 4\Delta_5 + \Delta_4 + \omega_4 + 4\Delta_5, 5\omega_5 + \omega_4 + 3\omega_3)$ captures the remaining case. Both branches have three components. Table 3 captures the combined branches of a immediately following final 5-branch and branches **(A)** and **(B)**. This depends on whether the final 5-regular branch follows after the first (1), the second (2) or the third (3) component or in any combination of them.

Branch-type	components	combined branch	upper bound
(A)	(1)	$(\{6\omega_5 + \omega_4 + 5\Delta_5\}^2, 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + \omega_4 + 3\Delta_4 + \Delta_5, 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + 4\omega_4 + \omega_3 + \Delta_5)$	$\mathcal{O}^*(1.0912^K)$
(A)	(2)	$(5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5), \{6\omega_5 + \omega_4 + 5\Delta_5\}^2, 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + 4\omega_4 + \omega_3 + \Delta_5)$	$\mathcal{O}^*(1.1094^K)$
(A)	(3)	$(5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5), 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + \omega_4 + 3\Delta_4 + \Delta_5, \{6\omega_5 + \omega_4 + 5\Delta_5 + \omega_3\}^2)$	$\mathcal{O}^*(1.1175^K)$
(B)	(1)	$(\{6\omega_5 + \omega_4 + 5\Delta_5\}^2, 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + \omega_4 + 4\Delta_5, 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + 4\omega_4 + \omega_3 + 3\omega_3)$	$\mathcal{O}^*(1.0894^K)$
(B)	(2)	$(5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5), \{6\omega_5 + \omega_4 + 5\Delta_5\}^2, 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + 4\omega_4 + \omega_3 + 3\omega_3)$	$\mathcal{O}^*(1.1052^K)$
(B)	(3)	$(5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5), 5\Delta_5 + \omega_5 + (\Delta_4 - \Delta_5) + \omega_4 + 4\Delta_5, \{6\omega_5 + \omega_4 + 5\Delta_5 + 3\omega_3\}^2)$	$\mathcal{O}^*(1.1159^K)$
(A) or (B)	(1)(2)/(1)(3) /(2)(3)	$(\{6\omega_5 + \omega_4 + 5\Delta_5\}^4, \omega_5 + 4\Delta_5 + \Delta_4)$	$\mathcal{O}^*(1.1126^K)$
(A) or (B)	(1)(2)(3)	$(\{6\omega_5 + \omega_4 + 5\Delta_5\}^6)$	$\mathcal{O}^*(1.0936^K)$

Table 3. The second column indicates after which components a final 5-regular branch immediately follows.

We would like to comment the recurrences in the third and sixth row. Here we get a reduction of ω_3 and $3\omega_3$ in addition to $5\omega_4 + \omega_4$ from v, v_1, \dots, v_5 . This additional amount comes from clauses C such that $|C \cap \{v, v_1, \dots, v_5\}| = 1$. Especially in the first case due to $\text{scp } N[v]$ is not component and thus at least one further variable must be deleted.

Proposition 8. *Let $v \in V(F)$ be the variable chosen for branching by Alg. 1 such that $\#_2(v) = 5$. Assume v induces a solution to equation (1) such that it is different from $k_{13} = 1, k_{15} = 4; k_{15} = 5; k_{25} = 1, k_{15} = 3; k_{14} = 1, k_{15} = 4$ (\star). If a 5-regular branch follows in one component we have at least a $(\{3\omega_5 + \omega_4 + 5\Delta_5\}^2, \omega_5 + 3\Delta_5 + 2\Delta_4)$ -branch and if it follows in both a $(\{3\omega_5 + \omega_4 + 5\Delta_5\}^4)$ -branch.*

Proof. If a component is followed by a final 5-regular branch the least amount we get by reduction from $N(v)$ is $\omega_5 + \omega_4$. This refers to the case $k_{35} = 1$ and $k_{24} = 1$ which follows analogously from Section C.1.

The least reduction from $N(v)$ without a following final 5-regular branch can be found as follows: Consider any solution of equation (1) except the ones in (\star). Among them find one which minimizes

$$\Delta_3 k_{13} + \Delta_4 k_{14} + \Delta_5 k_{15} + (\Delta_4 + \Delta_3) k_{24} + (\Delta_5 + \Delta_4) k_{25} + (\Delta_5 + \Delta_4 + \Delta_3) k_{35} \quad (3)$$

We can assume $k_{24} = 0$ as we have $\Delta_4 + \Delta_3 > \Delta_5 + \Delta_4$. As we are excluding (\star) we must have $k_{15} \leq 4$. If $k_{15} = 4$ we conclude that either $k_{14} = 1$ or $k_{13} = 1$. Both solutions are forbidden (see (\star)). Thus we must have $k_{15} \leq 3$.

If $k_{35} = 1$ then there is a better solution as $\Delta_5 + 2\Delta_4 < \Delta_5 + \Delta_4 + \Delta_3$: set $k'_{35} = 0, k'_{15} = k_{15} + 1, k'_{14} = k_{14} + 2$ and keep the other coefficients. Note that in this case we must have $k_{15} \leq 2$ which assures that the new solution is different from the ones in (\star). Therefore it follows that $k_{35} = 0$.

Now suppose $k_{25} = 2$, then $k_{15} = 1$ holds. But then $k'_{25} = k_{25} - 1, k'_{15} = k_{15} + 1, k'_{14} = 1$ is a no worse solution. Thus $k_{25} \leq 1$. If $k_{25} = 1$ then with $k_{15} = 2$ and $k_{14} = 1$ this is minimal (since $k_{15} = 3$ is forbidden (\star)).

Suppose $k_{25} = 0$, then clearly the best solution is $k_{15} = 3$ and $k_{14} = 2$. Both solutions provide a reduction of $3\Delta_5 + 2\Delta_4$ which is minimal.

Now we analyze a final 5-regular branch and a branch different from (\star) satisfying equation (1). If the final 5-regular branch follows in only one component then we have at least a $(\{3\omega_5 + \omega_4 + 5\Delta_5\}^2, \omega_5 + 3\Delta_5 + 2\Delta_4)$ -branch in the combined analysis. If it follows in both then a $(\{3\omega_5 + \omega_4 + 5\Delta_5\}^4)$ -branch upper bounds correctly. \square

Due to Proposition 8 we can upper bound the 5-regular branches whose predecessors are different from $k_{13} = 1, k_{15} = 4; k_{15} = 5; k_{25} = 1, k_{15} = 3; k_{14} = 1, k_{15} = 4$ by $\mathcal{O}^*(1.1171^K)$ in their combined analysis.

F.2 Analysis of the 6- 4- and 3-phase in the Combined Approach

Here we provide the run times under $\tilde{\gamma}(F)$ for the cases we did not consider in Section 5. The run time has been estimated with respect to $\tilde{\gamma}(F)$. Names will refer to the corresponding ones in the analysis of Alg. 1.

Non-regular Branches In Table 4 we find the derived recurrences for each priority of **HP** if we have a non-regular branch. You can find them together with their run times. Priority 3 is not considered as the 5-phase has been analyzed in Sections 5 and F.1.

Regular Branches Table 5 captures the run times of any internal 6, 5 or 4-regular branch. Table 6 considers final 4 or 6-regular branches together with their preceding branches. The case where we have chosen v due to priority 7 such that v has a 3-path with $u_0 \neq u_l$ is treated separately.

Priorities	branch	upper bound
Priority 1	$(7, 7)$	$\mathcal{O}^*(1.1042^K)$
Priority 2	$(\{\omega_6 + 5\Delta_6 + \Delta_5\}^2)$	$\mathcal{O}^*(1.118^K)$
Priority 4	$(\{2\omega_4 + 4\Delta_4\}^2)$	$\mathcal{O}^*(1.102^K)$
Priority 5	$(\{3\omega_4 + 2\omega_3\}^2)$	$\mathcal{O}^*(1.1099^K)$
Priority 6	$(2\omega_4 + 2\omega_3 + 2\Delta_4, \omega_4 + 2\omega_3 + 2\Delta_4)$	$\mathcal{O}^*(1.1143^K)$
Priority 7	$(\omega_4 + \omega_3 + 5\Delta_4, \omega_4 + \omega_3 + 3\Delta_4)$	$\mathcal{O}^*(1.1172^K)$
	$(\{\omega_4 + 3\omega_3 + 3\Delta_4\}^2)$	$\mathcal{O}^*(1.1^K)$
	$(2\omega_4 + \omega_3 + 3\Delta_4, \omega_4 + \omega_3 + 3\Delta_4)$	$\mathcal{O}^*(1.1165^K)$
Priority 8	$(\{2\omega_4 + 2\omega_3 + 2\Delta_4\}^2)$	$\mathcal{O}^*(1.1^K)$
Priority 9	$(8\omega_3, 6\omega_3)$	$\mathcal{O}^*(1.1105^K)$
Priority 10	$(8\omega_3, 6\omega_3)$	$\mathcal{O}^*(1.1105^K)$
	$(4\omega_3, 10\omega_3)$	$\mathcal{O}^*(1.118^K)$

Table 4. The non-regular cases

case	branch	upper bound
Internal 6-regular	$(3\omega_6, \omega_6 + 6\Delta_6)$	$\mathcal{O}^*(1.0978^K)$
	$(\{3\omega_6\}^2)$	$\mathcal{O}^*(1.0802^K)$
Internal 5-regular	$(3\omega_5, \omega_5 + 5\Delta_5)$	$\mathcal{O}^*(1.1112^K)$
	$(\{3\omega_5\}^2)$	$\mathcal{O}^*(1.0974^K)$
Internal 4-regular	$(5\omega_4, \omega_4 + 4\Delta_4)$	$\mathcal{O}^*(1.103^K)$
	$(\{5\omega_4\}^2)$	$\mathcal{O}^*(1.079^K)$

Table 5. Internal h -regular cases ($h \in \{4, 5, 6\}$) and their upper bounds.

3-path Finally we consider the case when a variable chosen to priority 7 has a 3-path with $u_0 \neq u_{l+1}$. The cases *a*) $(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \omega_4 + \omega_3 + 3\Delta_4)$, *b*) $(\omega_4 + \omega_3 + 5\Delta_4, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$ and *c*) $(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$ are upper bounded by $\mathcal{O}^*(1.1152^K)$, $\mathcal{O}^*(1.1159^K)$ and $\mathcal{O}^*(1.1147^K)$. Table 7 captures the branches together with their run times in the combined algorithm if for all $y \in N(v)$ we have $\#_2(y) = 3$.

We have a $(\{2\omega_4 + 7\omega_3 + 4\Delta_4\}^2, \omega_4 + 4\omega_3)$ -branch for case α) which $\mathcal{O}^*(1.1132^K)$ properly upper bounds. We also have a $(\omega_4 + 4\omega_3 + 2\Delta_4, \{2\omega_4 + 5\omega_3 + 4\Delta_4\}^2)$ -branch for case β) such that it is upper bounded by $\mathcal{O}^*(1.1142^K)$.

Preceding branch	branch	upper bound
Final 6-regular Branch		
Any 6-phase branch	$(\{3\omega_6 + \omega_4 + 6\Delta_6\}^2, \omega_6 + 6\Delta_6)$ $(\{3\omega_6 + \omega_4 + 6\Delta_6\}^4)$	$\mathcal{O}^*(1.11^K)$ $\mathcal{O}^*(1.105^K)$
Final 4-regular Branch		
Internal 4-regular	$(\{6\omega_4 + 4\Delta_4\}^2, \omega_4 + 4\Delta_4)$ $(\{6\omega_4 + 4\Delta_4\}^4)$	$\mathcal{O}^*(1.1115^K)$ $\mathcal{O}^*(1.1003^K)$
Priorities 4,5 and 8 Cases 2,3(a),4 of Priority 6	$(\{3\omega_4 + 8\Delta_4\}^2, 2\omega_4 + 4\Delta_4)$ $(\{3\omega_4 + 8\Delta_4\}^4)$	$\mathcal{O}^*(1.1115^K)$ $\mathcal{O}^*(1.117^K)$
Priority 6 Case 1 Case 5,3(b) b) and c) of the analysis Case 3b), case a) of the analysis Case 5, case a) of the analysis	$(\{3\omega_4 + 2\omega_3 + 4\Delta_4\}^2)$ $(\{2\omega_4 + 2\omega_3 + 2\Delta_4, \{3\omega_4 + 4\omega_3 + 4\Delta_4\}^2\}^2)$ b) $(\{3\omega_4 + 4\omega_3 + 4\Delta_4\}^4)$ c) $(\{3\omega_4 + 4\omega_3 + 5\Delta_4\}^2, \omega_4 + 2\omega_3 + 2\Delta_4)$ $(\{3\omega_4 + 4\omega_3 + 5\Delta_4\}^2, 2\omega_4 + 2\omega_3)$	$\mathcal{O}^*(1.07^K)$ $\mathcal{O}^*(1.109^K)$ $\mathcal{O}^*(1.1143^K)$ $\mathcal{O}^*(1.1151^K)$ $\mathcal{O}^*(1.1145^K)$
Priority 7 Case of a 4-path Case b) Case a') Case c') Case a'') Case c'') Case of a 3-path with $u_0 = u_{l+1}$	$(2\omega_4 + \omega_3 + 3\Delta_4, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$ $(\{4\omega_4 + 3\omega_3 + 4\Delta_4\}^2, \omega_4 + \omega_3 + 3\Delta_4)$ $\{4\omega_4 + 3\omega_3 + 4\Delta_4\}^2, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$ $(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \omega_4 + \omega_3 + 3\Delta_4)$ $(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2, \{3\omega_4 + 3\omega_3 + 4\Delta_4\}^2)$ similar to priorities 4,5 and 8	$\mathcal{O}^*(1.1155^K)$ $\mathcal{O}^*(1.1156^K)$ $\mathcal{O}^*(1.115^K)$ $\mathcal{O}^*(1.1152^K)$ $\mathcal{O}^*(1.1147^K)$

Table 6. The final h -regular cases ($h \in \{4, 6\}$) and their combined analysis

$\#_2(u_0), \#_2(u_{l+1})$	left component	right component	both components
$\#_2(u_0) = 3$ $\#_2(u_{l+1}) = 3$ upper bounds	case α instead	$(\omega_4 + 6\omega_3,$ $\{2\omega_4 + 6\omega_3 + 4\Delta_4\}^2)$ $\mathcal{O}^*(1.1075^K)$	$(\{2\omega_4 + 6\omega_3 + 4\Delta_4\}^2,$ $\{2\omega_4 + 6\omega_3 + 4\Delta_4\}^2)$ $\mathcal{O}^*(1.1136^K)$
$\#_2(u_0) = 3$ $\#_2(u_{l+1}) = 4$ upper bounds	$(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2,$ $\omega_4 + 4\omega_3)$ $\mathcal{O}^*(1.1136^K)$	$(\omega_4 + 5\omega_3 + \Delta_4,$ $\{2\omega_4 + 5\omega_3 + 4\Delta_4\}^2)$ $\mathcal{O}^*(1.1138^K)$	$(\{3\omega_4 + 5\omega_3 + 4\Delta_4\}^2,$ $\{2\omega_4 + 5\omega_3 + 4\Delta_4\}^2)$ $\mathcal{O}^*(1.1143^K)$
$\#_2(u_0) = 4$ $\#_2(u_{l+1}) = 4$ upper bounds	$(\{4\omega_4 + 4\omega_3 + 4\Delta_4\}^2,$ $\omega_4 + 4\omega_3)$ $\mathcal{O}^*(1.1088^K)$	case β instead	$(\{4\omega_4 + 4\omega_3 + 4\Delta_4\}^2,$ $\{2\omega_4 + 4\omega_3 + 4\Delta_4\}^2)$ $\mathcal{O}^*(1.1088^K)$

Table 7.

